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A FUNCTIONAL EQUATION CHARACTERIZING HOMOGRAPHIC FUNCTIONS

JANUSZ MATKOWSKI

Abstract. Some functional equations related to homographic functions and their characterization are presented.

1. Introduction

If $I \subset \mathbb{R}$ is an interval and $f: I \rightarrow \mathbb{R}$ is a homographic function

$$f(x) = \frac{ax + b}{cx + d}, \quad x \in I,$$

($ad - bc \neq 0$) then, it is easy to verify that

$$(*) \quad \left(\frac{f(x) - f(y)}{x - y} \right)^2 = f'(x)f'(y), \quad x, y \in I, \quad x \neq y,$$

cf. [1] where this equation has appeared in a problem related to convex functions.

Replacing here $\sqrt{f'}$ by an arbitrary function g we get the functional equation

$$\frac{f(x) - f(y)}{x - y} = g(x)g(y), \quad x, y \in I, \quad x \neq y,$$

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with two unknown functions f and g . We show that, without any regularity assumptions, this equation characterizes the homographic function and their derivative (Corollary 1).

In Section 1, we consider the functional equation

$$\frac{f(x) - f(y)}{x - y} = pg(x)g(y), \quad x, y \in X, \quad x \neq y,$$

assuming that X is an arbitrary subset of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ such that $\text{card } X \geq 3$ and $f, g: X \rightarrow \mathbb{K}$ are the unknown functions. Theorem 1 gives the general solution for $p = 1$, and Corollary 1 for $p \neq 0$.

In Section 2, Theorem 2 describes the general solution of the functional equation with four unknown functions

$$\frac{f(x) - F(y)}{x - y} = g(x)G(y), \quad x, y \in X, \quad x \neq y.$$

A remark on a general solution of the functional equation

$$\frac{f(x) - F(y)}{h(x) - h(y)} = g(x)G(y), \quad x, y \in X, \quad x \neq y,$$

with five unknown functions f, F, g, G, h defined on an arbitrary set such X such that $\text{card } X \geq 3$ ends up the paper.

Another, completely different, approach in a characterization of the homographic functions, more closer to the invariance of double ratio of four points, is implicitly given in [1].

2. Functional equation with two functions

The main result of this section reads as follows:

THEOREM 1. *Let $X \subset \mathbb{K}$ be a set such that $\text{card } X \geq 3$. The functions $f, g: X \rightarrow \mathbb{K}$ satisfy the functional equation*

$$(1) \quad \frac{f(x) - f(y)}{x - y} = g(x)g(y), \quad x, y \in X, \quad x \neq y,$$

if, and only if, either

f is an arbitrary constant and g is the zero function;

or

$$f(x) = \frac{ax+b}{cx+d}, \quad g(x) = \frac{1}{cx+d}, \quad x \in X,$$

for some $a, b, c, d \in \mathbb{K}$ such that $ad - bc = 1$.

PROOF. Assume that $f, g: X \rightarrow \mathbb{R}$ satisfy equation (1).

If there is $y \in X$ such that $g(y) = 0$, then, by (1), $f(x) = f(y)$ for all $x \in X$, that f is constant. It follows that $g(x) = 0$ for all $x \in X$. Obviously, if f is constant and $g = 0$ then equation (1) is satisfied.

Now we can assume that $g(x) \neq 0$ for all $x \in X$. From (1) we have

$$f(x) - f(y) = g(x)g(y)(x - y), \quad x, y \in X, \quad x \neq y.$$

Since $f(x) - f(y) = [f(x) - f(z)] + [f(z) - f(y)]$, we hence get

$$g(x)g(y)(x - y) = g(x)g(z)(x - z) + g(z)g(y)(z - y)$$

for all $x, y, z \in X$, $x \neq y \neq z \neq x$, or equivalently, dividing both sides by $g(x)g(y)g(z)$,

$$\frac{x - y}{g(z)} = \frac{x - z}{g(x)} + \frac{z - y}{g(y)}, \quad x, y, z \in X, \quad x \neq y \neq z \neq x,$$

whence, for all $x, y, z \in X$, $x \neq y \neq z \neq x$,

$$\frac{1}{g(z)} = \frac{1}{x - y} \left[\left(\frac{1}{g(x)} - \frac{1}{g(y)} \right) z + \left(\frac{x}{g(x)} - \frac{y}{g(y)} \right) \right].$$

Since the right side does not depend on z and, by assumption, $\text{card } X \geq 3$, it follows that there are $c, d \in \mathbb{K}$ such that

$$\frac{1}{g(z)} = cz + d, \quad z \in X,$$

whence

$$(2) \quad g(x) = \frac{1}{cx + d}, \quad x \in X.$$

Setting this function into equation (1) we get

$$f(x) = \frac{1}{cy + d} \frac{x - y}{cx + d} + f(y), \quad x, y \in X, \quad x \neq y,$$

whence, as the right side does not depend on y , we conclude that

$$(3) \quad f(x) = \frac{ax + by}{cx + d}, \quad x \in X,$$

for some $a, b \in \mathbb{K}$.

Substituting the functions (2) and (3) into (1) we obtain

$$\frac{f(x) - f(y)}{x - y} = \frac{ad - bc}{(cx + d)(cy + d)} = (ad - bc)g(x)g(y), \quad x, y \in X, \quad x \neq y,$$

which implies that $ad - bc = 1$. This completes the proof. \square

COROLLARY 1. *Let $X \subset \mathbb{K}$ be a set such that $\text{card } X \geq 3$ and let $p \in \mathbb{K} \setminus \{0\}$ be fixed. The functions $f, g: X \rightarrow \mathbb{K}$ satisfy the functional equation*

$$(4) \quad \frac{f(x) - f(y)}{x - y} = pg(x)g(y), \quad x, y \in X, \quad x \neq y,$$

if, and only if, either

f is an arbitrary constant and g is the zero function;

or

$$f(x) = \frac{ax + b}{cx + d}, \quad g(x) = \frac{1}{cx + d}, \quad x \in X,$$

for some $a, b, c, d \in \mathbb{K}$ such that

$$ad - bc = p.$$

PROOF. It is enough to apply Theorem 1 with f replaced by f/p . \square

COROLLARY 2. *Let $I \subset \mathbb{R}$ be an interval. A differentiable function $f: I \rightarrow \mathbb{R}$ satisfies equation (*):*

$$\left(\frac{f(x) - f(y)}{x - y} \right)^2 = f'(x)f'(y), \quad x, y \in I, \quad x \neq y,$$

if, and only if, either f constant or

$$f(x) = \frac{ax + b}{cx + d}, \quad x \in I,$$

for some $a, b, c, d \in \mathbb{K}$ such that

$$ad - bc \neq 0.$$

PROOF. Obviously, f is constant iff $f'(x) = 0$ for some $x \in I$. Therefore it is enough consider the case when f' is of the constant sign in I . Without any loss of generality, we can assume that f' is positive in I . Then, clearly,

$$\frac{f(x) - f(y)}{x - y} > 0, \quad x, y \in J, \quad x \neq y.$$

Hence, setting $g := \sqrt{f'}$ we get the functional equation (1). Applying Theorem 1 we obtain

$$f(x) = \frac{ax + b}{cx + d}, \quad x \in I.$$

Now it easy to verify that f satisfies equation (*). □

3. Functional equation with four unknown functions

Applying Theorem we shall prove the following

THEOREM 2. *Let $X \subset \mathbb{K}$ be a set such that $\text{card } X > 3$. The functions $f, F, g, G: X \rightarrow \mathbb{K}$ satisfy the functional equation*

$$(5) \quad \frac{f(x) - F(y)}{x - y} = g(x)G(y), \quad x, y \in X, \quad x \neq y,$$

if, and only if, one of the following cases occurs:

(i) *for some $a, b, c, d, m \in \mathbb{C}$ such that $ad - bc \neq 0$,*

$$f(x) = F(x) = \frac{ax + b}{cx + d}, \quad g(x) = \frac{G(x)}{ac - bd} = \frac{1}{cx + d}, \quad x \in X;$$

(ii) *the functions f and F are constant, $f = F$, $g(x) = 0$ for all $x \in X$ and G is arbitrary;*

(iii) *the functions f and F are constant, $f = F$, $G(x) = 0$ for all $x \in X$ and g is arbitrary.*

PROOF. Assume that $g(x)G(x) \neq 0$ for all $x \in X$. Interchanging x and y in (5) and we get

$$\frac{f(y) - F(x)}{y - x} = g(y)G(x), \quad x, y \in X, \quad x \neq y.$$

Dividing the respective sides of these two equations we obtain

$$(6) \quad \frac{f(x) - F(y)}{F(x) - f(y)} = \frac{m(x)}{m(y)}, \quad x, y \in X, \quad x \neq y,$$

where

$$m(x) := \frac{g(x)}{G(x)}, \quad x \in X.$$

Setting an arbitrary chosen $y = y_0$ in (6) we get

$$(7) \quad F(x) = \frac{\alpha f(x) + \beta}{m(x)} + \gamma, \quad x \in X \setminus \{y_0\},$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$. From (5) and (6) we obtain

$$\frac{f(x) - \frac{\alpha f(y) + \beta}{m(y)} - \gamma}{\frac{\alpha f(x) + \beta}{m(x)} + \gamma - f(y)} = \frac{m(x)}{m(y)}, \quad x, y \in X \setminus \{y_0\}, \quad x \neq y,$$

whence, after simplification,

$$(8) \quad f(x) [m(y) - \alpha] = [\alpha + \gamma - f(y)] m(x) + \gamma m(y) + 2\beta$$

for all $x, y \in X \setminus \{y_0\}, x \neq y$.

Consider the case when

$$m(x) = \alpha, \quad x \in X \setminus \{y_0\}.$$

Of course $\alpha \neq 0$. In this case (8) implies that $\gamma = -\frac{\beta}{\alpha}$ and, from (7),

$$F(x) = f(x), \quad x \in X \setminus \{y_0\}.$$

Thus from (5) we get the functional equation

$$(9) \quad \frac{f(x) - f(y)}{x - y} = g(x)G(y), \quad x, y \in X \setminus \{y_0\},$$

with four unknown functions. The symmetry of the left-hand side implies that

$$\frac{g(x)}{G(x)} = \frac{g(y)}{G(y)} = a, \quad x, y \in X \setminus \{y_0\}, \quad x \neq y,$$

whence

$$G(y) = \frac{1}{a}g(y), \quad y \in X \setminus \{y_0\},$$

and from (9) we get

$$\frac{f(x) - f(y)}{x - y} = \frac{1}{\alpha}g(x)g(y), \quad x, y \in X \setminus \{y_0\}, \quad x \neq y.$$

Repeating this reasoning with y_0 replaced by y'_0 , $y'_0 \neq y_0$, we conclude that

$$\frac{f(x) - f(y)}{x - y} = \frac{1}{\alpha}g(x)g(y), \quad x, y \in X \setminus \{y'_0\}, \quad x \neq y.$$

Both equations imply that

$$\frac{f(x) - f(y)}{x - y} = \frac{1}{\alpha}g(x)g(y), \quad x, y \in X, \quad x \neq y.$$

Applying Corollary 1 with $p := \frac{1}{\alpha}$ we obtain the "if" part of our result.

To finish the proof it is enough to show that the function m must coincide with the constant α . For an indirect argument assume that there is $y_1 \in X$ such that

$$m(y_1) \neq \alpha.$$

Then, from (8) we get

$$(10) \quad f(x) = Am(x) + B, \quad x \in X \setminus \{y_0\},$$

for some $A, B \in \mathbb{R}$.

We shall show that in this case m and f are constant functions in $X \setminus \{y_0\}$. Assume first that $A = 0$. Then $f(x) = B$ for all $x \in X \setminus \{y_0\}$ and from (7)

$$(11) \quad F(x) = \frac{l}{m(x)} + \gamma, \quad x \in X \setminus \{y_0\},$$

where $l := \alpha B + \beta$. If l were zero then we would have $F(x) = \gamma$ for all $x \in X \setminus \{y_0\}$, and from (7),

$$\frac{B - \gamma}{\gamma - B} = \frac{m(x)}{m(y)}, \quad x, y \in X \setminus \{y_0\}, \quad x \neq y,$$

that is $m(y) = -m(x)$ for all $x, y \in X \setminus \{y_0\}$, $x \neq y$. This is impossible as the set $X \setminus \{y_0\}$ has at least three points and m cannot disappear at any point. Thus $l \neq 0$. Put $C := B - \gamma$. Setting the function (11) into (6) we get

$$\frac{C - \frac{l}{m(y)}}{\frac{l}{m(x)} - C} = \frac{m(x)}{m(y)}, \quad x, y \in X, \quad x \neq y,$$

whence

$$C[m(x) + m(y)] = 2l, \quad x, y \in X \setminus \{y_0\}, \quad x \neq y.$$

Since $l \neq 0$, it follows that m is a constant function in $X \setminus \{y_0\}$.

Now we can assume that $A \neq 0$. Setting f given by (10) into (8), after simple calculations, we get

$$m(x)[2Am(y) - \alpha A - \alpha + \beta - \gamma] = \gamma m(y) - \beta m(y) + \alpha B + 2\beta, \quad x, y \in X \setminus \{y_0\}.$$

This equality implies that m is a constant function. Indeed, if $(X \setminus \{y_0\}) \ni x \rightarrow m(x)$ were not constant then we would have

$$2Am(y) - \alpha A - \alpha + \beta - \gamma = 0, \quad y \in X \setminus \{y_0\},$$

and, as $A \neq 0$,

$$m(y) = \frac{\alpha A + \alpha - \beta + \gamma}{2A}, \quad y \in X \setminus \{y_0\},$$

that is a contradiction.

Thus the functions m and, by (10), f are constant in $X \setminus \{y_0\}$. In view of (7) also F is constant. Then $D := f - F$ is a constant and, from (5), $D \neq 0$. From (6) we get

$$m(y) = -m(x), \quad x, y \in X \setminus \{y_0\}, \quad x \neq y,$$

that is impossible.

Now consider the case when $g(x)G(x) = 0$ for some $x \in X$.

Let $Z_G := \{x \in X: G(x) = 0\}$. Suppose that $Z_G \neq \emptyset$ and assume that $y_0 \in Z_G$. Then, by equation (5), $f(x) = F(y_0) =: \gamma$ for all $x \in X$, so f is

a constant function. Similarly, if there is $x_0 \in Z_g$ then $F(x) = f(x_0)$ for all $x \in X$. Moreover $F = f$ on $Z_G \cup Z_g$. From (5) we have

$$\frac{\gamma - F(y)}{x - y} = g(x)G(y), \quad x, y \in X, \quad x \neq y,$$

and, after interchanging a and y ,

$$\frac{\gamma - F(y)}{y - x} = g(y)G(x), \quad x, y \in X, \quad x \neq y$$

Putting $Y := X \setminus (Z_G \cup Z_g)$ and

$$m(x) := \frac{g(x)}{G(x)}, \quad x \in Y,$$

we hence get

$$\frac{\gamma - F(y)}{F(x) - \gamma} = \frac{m(x)}{m(y)}, \quad x, y \in Y, \quad x \neq y,$$

that is

$$m(x) [F(x) - \gamma] = m(y) [\gamma - F(y)], \quad x, y \in Y, \quad x \neq y,$$

whence, for a constant β ,

$$F(x) = \frac{\beta}{m(x)} + \gamma \quad \text{for all } x \in Y, \quad \text{and} \quad F(x) = \frac{-\beta}{m(x)} + \gamma, \quad x \in Y.$$

It follows that $\beta = 0$ and, consequently, $F(x) = \gamma$ for all $x \in Y$. Thus we have shown that F and f are constant and equal on X . Hence, setting these functions into (5) we get $g(x)G(y) = 0$ for all $x, y \in X, x \neq y$. It follows that either g or G is the zero function. The proof is completed. \square

4. Remark on a functional equation with five unknown functions

Applying Theorem 2 we get the following

REMARK 1. *Let X be an arbitrary set such that $\text{card } X > 3$. The functions $f, F, g, G, h: X \rightarrow \mathbb{K}$ where h is one-to-one satisfy the functional equation*

$$\frac{f(x) - F(y)}{h(x) - h(y)} = g(x)G(y), \quad x, y \in X, \quad x \neq y,$$

if, and only if, one of the following cases occurs:

(i) for some $a, b, c, d, m \in \mathbb{C}$ such that $ad - bc \neq 0$,

$$f(x) = F(x) = \frac{ah(x) + b}{ch(x) + d}, \quad g(x) = \frac{G(x)}{ac - bd} = \frac{1}{ch(x) + d}, \quad x \in X,$$

- (ii) the functions f and F are constant, $f = F$, $g(x) = 0$ for all $x \in X$ and G is arbitrary;
- (iii) the functions f and F are constant, $f = F$, $G(x) = 0$ for all $x \in X$ and g is arbitrary.

To get this remark it is enough to apply Theorem 2 to the functional equation

$$\frac{(f \circ h^{-1})(x) - (F \circ h^{-1})(y)}{x - y} = (g \circ h^{-1})(x) (G \circ h^{-1})(y),$$

$$x, y \in h(X), \quad x \neq y.$$

Reference

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